Quantitative Finance

Conditional Heteroskedastic Models

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Prof Engle je nositelem Nobelovy ceny za ekonomii pro rok 2003
Miloslav Vosvrda; 23.3.2004
Recent Awards

*The Bank of Sweden Prize in Economic Sciences*

in

*Memory of Alfred Nobel 2003*

for

methods of analyzing economic time series

with time-varying volatility (ARCH)
The Pivotal Interest in almost all financial applications is

The Predictability of Price Changes
The Pivotal Requirement
in almost all
financial applications
is
Any Volatility Model
Must Be Capable to
Forecast the Volatility
Financial Time Series

The volatile behavior in financial markets is usually referred to as the “volatility”. Volatility has become a very important concept in different areas in financial theory and practice. Volatility is usually measured by variance, or standard deviation. The financial markets are sometimes more volatile, and sometimes less active. Therefore a conditional heteroskedasticity and stylized facts of the volatility behavior observed in financial time series are typical features for each financial market.
ARCH, GARCH

This one is a class of (generalized) autoregressive conditional heteroskedasticity models which are capable of modeling of time varying volatility and capturing many of the stylized facts of the volatility behavior observed in financial time series.
Stylized Facts

1) Non-Gaussian, heavy-tailed, and skewed distributions. The empirical density function of returns has a higher peak around its mean, but fatter tails than that of the corresponding normal distribution. The empirical density function is tailer, skinnier, but with a wider support than the corresponding normal density. Figure

2) Volatility clustering (ARCH-effects). Figure

3) Temporal dependence of the tail behavior.
The graph shows two functions:

- Red line: \( \text{dnorm}(x, 0, 0.7) \)
- Blue dashed line: \( \text{dt}(x, 3) \)
4) Short- and long-range dependence.

5) Asymmetry-Leverage effects. There is evidence that the distribution of stock returns is slightly negatively skewed. Agents react more strongly to negative information than to positive information. Figure
EGARCH(1,1)

returns $r_n$

time

$54.403$ $10.126$
Capital Market Price Models

• The Random Walk Model (RWM)
• Martingale (MGL)
• Conditional Heteroscedastic Model (CHM)
  – Models of the first category
    • ARCH
    • GARCH
      – IGARCH
      – EGARCH
      – CHARMA
    • ARCH-M
    • GARCH-M
  – Models of the second category
    • Stochastic volatility model
The Random Walk Model (RWM)

\[ S_t \] the price observed at the beginning of time \( t \)
\[ \varepsilon_t \] an error term with \( E(\varepsilon_t) = 0 \) and \( \text{Var}(\varepsilon_t) = \sigma^2_\varepsilon \)

values of the independent of each other

\[ S_t = S_{t-1} + \varepsilon_t \]
\[ S_t - S_{t-1} = \varepsilon_t \]
\[ S_t = \sum_{j=1}^{t} \varepsilon_j \]

RWM was first hypothesis about how financial prices move. Fama (1965) compiled EMH.
A Prediction by the RWM

The 1-step ahead forecast at the forecast origin $h$ is

$$\hat{S}_h (1) = E \left( S_{h+1} \mid S_h, S_{h-1}, \ldots \right) = S_h$$

The 2-step ahead forecast at the forecast origin $h$ is

$$\hat{S}_h (2) = E \left( S_{h+2} \mid S_h, S_{h-1}, \ldots \right) =$$

$$= E \left( S_{h+1} + \varepsilon_{h+2} \mid S_h, S_{h-1}, \ldots \right) = \hat{S}_h (1) = S_h$$

For any $l > 0$ forecast horizon we have

$$\hat{S}_h (l) = S_h$$

Therefore, the RWM is not mean-reverting
Forecast Error

The l-step ahead forecast error is

\[ e_h (l) = \varepsilon_{h+l} + \cdots + \varepsilon_{h+1} \]

So that

\[ \text{Var} \left[ e_h (l) \right] = l \sigma_\varepsilon^2 \]

which diverges to infinity as \( l \to \infty \)

The RWM is not predictable.
RWM with drift

RWM with drift for the price $S_t$ is

$$S_t = \mu + S_{t-1} + \varepsilon_t$$

where

$$\mu = E \left( S_t - S_{t-1} \right)$$

Then

$$S_t = S_0 + t \mu + \sum_{j=1}^{t} \varepsilon_j$$

and

$$Var \left( \sum_{i=1}^{t} \varepsilon_i \right) = t\sigma_{\varepsilon}^2$$

Therefore the conditional standard deviation $\sqrt{t\sigma_{\varepsilon}^2}$ grows slower than the conditional expectation.
Le Roy’s critique of RWM

The critique of RWM by Le Roy (1989) led to some serious questions about RWM as the theoretical model of financial markets. The assumption that price changes are independent was found to be too restrictive. Therefore was, after broad discussion, suggested the following model

\[ r_t = \frac{S_{t+1} - S_t + D_t}{S_t} \]

where
- \( r \) is a return
- \( D \) is a dividend
We assume that $E(r_t | \mathcal{F}_t) = r$ is a constant.

Taking expectations at time $t$ of the both sides,

$$r_t = \frac{S_{t+1} - S_t + D_t}{S_t}$$

and we get

$$S_t = \frac{E(S_{t+1} + D_t | \mathcal{F}_t)}{1 + r}$$

We assume that reinvesting of dividends is

$$x_t = h_t \cdot S_t$$ and $$h_{t+1} \cdot S_{t+1} = h_t \cdot (S_{t+1} + D_t)$$
Thus

\[ E(x_{t+1} \mid \mathcal{F}_t) = E(h_{t+1} \cdot S_{t+1} \mid \mathcal{F}_t) = h_t \cdot (S_{t+1} + D_t) = \]

\[ = (1 + r) h_t \cdot S_t = (1 + r) x_t \]

i.e. that \( x_t \) is a martingale.

For \( r > 0 \), \( x_t \) is a submartingale, because

\[ E(x_{t+1} \mid \mathcal{F}_t) \geq x_t \]

For \( r < 0 \), \( x_t \) is a supermartingale, because

\[ E(x_{t+1} \mid \mathcal{F}_t) \leq x_t \]
Very Important Distinction

A stochastic process following a **RWM** is more restrictive than the stochastic process that follows a **martingale**.

A financial series is known to go through protracted both **quiet periods** and **periods of turbulence**. This type of behavior could be modeled by a process with **conditional variances**. Such a specification is consistent with a martingale, but not with RWM.
Martingale processes lead to non-linear stochastic processes that are capable of modeling higher conditional moments. Such models are called **Conditional Heteroskedastic Models**. These ones are very important for a modeling of the volatility. The volatility is an very important factor in options trading. The volatility means the conditional variance of underlying asset return

\[
\sigma_t^2 = \text{Var}(r_t | \mathcal{F}_{t-1}) = E\left( (r_t - \mu_t)^2 | \mathcal{F}_{t-1} \right), \text{ where } \mu_t = E\left( r_t | \mathcal{F}_{t-1} \right)
\]
Conditional Heteroskedastic Model (CHM)

Let \( r_t \) is a stationary ARMA\((p,q)\) model, i.e.,

\[
    r_t = \mu_t + \varepsilon_t
\]

\[
    \mu_t = E\left(r_t \mid \mathcal{F}_{t-1}\right)
\]

\[
    \mu_t = \mu_0 + \sum_{i=1}^{p} \phi_i \cdot r_{t-i} + \sum_{i=1}^{q} \theta_i \cdot \varepsilon_{t-i}
\]

\[
    \sigma_t^2 = \text{Var}\left(r_t \mid \mathcal{F}_{t-1}\right) = E\left((r_t - \mu_t)^2 \mid \mathcal{F}_{t-1}\right) = E\left(\varepsilon_t^2 \mid \mathcal{F}_{t-1}\right)
\]
We distinguish two categories of CHM

- **The first category**: An exact function to a governing of the evolution of the $\sigma_t^2$ is used
- **The second category**: A stochastic equation to a describing of the $\sigma_t^2$ is used

The models coming under the first category are models of type ARCH or GARCH. These models may catch three out of the five stylized features, namely 1), 2), and 4).

The models coming under the second category are models of type Stochastic Volatility Model.
ARCH model

ARCH model

The basic idea  \( r_t = \mu_t + \varepsilon_t \)

\[ \varepsilon_t = r_t - \mu_t \]

\[ = \sigma_t \cdot \eta_t \quad \text{innovations} \]

\[ \sigma_t^2 = \alpha_0 + \alpha_1 \varepsilon_{t-1}^2 + \cdots + \alpha_m \varepsilon_{t-m}^2 \]

\( \eta_t, \ E(\eta_t) = 0, \ Var(\eta_t) = 1 \)
\( \alpha_0 > 0, \alpha_1 \geq 0, \cdots, \alpha_m \geq 0 \)

\( \eta_t \sim N(0,1) \ or \ \eta_t \sim t(\cdot) \)
ARCH(1)

\[ \varepsilon_t = \sigma_t \cdot \eta_t \]
\[ \eta_t \sim N(0,1) \]
\[ \alpha_0 > 0, \alpha_1 > 0 \]

so that \[ \sigma_t^2 = \alpha_0 + \alpha_1 \varepsilon_{t-1}^2 \]

This process is stationary if, and only if, \[ \alpha_1 < 1. \]

Unconditional variance \( \sigma_{\varepsilon}^2 \) is equal \[ \alpha_0 / (1 - \alpha_1). \]

The fourth moment is finite if \[ 3\alpha_1^2 < 1. \]

The kurtosis is so given by \[ 3(1 - \alpha_1^2) / (1 - 3\alpha_1^2). \]
ARCH(1)

\[ r_n := \mu_n + \varepsilon_n \cdot \sqrt{\alpha_0 + \alpha_1 \cdot (\varepsilon_{n-1})^2} \]

\[ \mu_n := \beta_0 + \beta_1 \cdot n \]

\[ \alpha_0 := 0.1 \quad \alpha_1 := 0.1 \quad \beta_0 := 0.02 \quad \beta_1 := 0.007 \]
ARCH(1)

returns

\( r_n - \mu_n \)

\( n \) time
$\alpha_0 + \alpha_1 (\varepsilon_{n-1})^2$

ARCH(1) Volatility

$\sigma \varepsilon$

volatility

$n$

time

$1 \times 10^3$
Autocorrelation Function

- $A_i$
- $\sigma_{\epsilon}$
- $\pm \sigma_{\epsilon}$
- $r_{- \text{mi}}$
- $(r_{- \text{mi}})^2$
- $\pm \text{Standard deviation}$

Graph showing the autocorrelation function with various lines representing different parameters and their standard deviations.
Partial Autocorrelation Function

- $P_i$
- $PP_i$
- $\sigma \varepsilon$
- $- \sigma \varepsilon$

- $r-mi$
- $(r-mi)^2$
- $+ Standard deviation$
- $- Standard deviation$
Simulated ARCH(1) errors

Simulated ARCH(1) volatility
Sample Quantiles:

min           1Q            median           3Q            max
-0.9662       -0.1044       0.005786       0.1096        1.225

Sample Moments:

mean          std          skewness        kurtosis
0.002704      0.2315       0.1384         7.016

Number of Observations: 500
The fourth moment \((rr, 0, 4)\) = 6.214

The fourth moment \((rrr, 0, 4)\) = \(2.402 \times 10^3\)

The third moment \((rr, 0, 3)\) = 1.85

The third moment \((rrr, 0, 3)\) = 108.796
Daily Stock Returns of FORD

Graph showing daily stock returns of FORD from 1984 to 1992.
The kurtosis exceeds 3, so that the unconditional distribution of $\varepsilon$ is fatter tailed than the normal.

Testing for ARCH effects. This test is constructed on a simple Lagrange Multiplier (LM) test. The null hypothesis is that: There are no ARCH effects, i.e.,

$$\alpha_1 = \cdots = \alpha_m = 0$$

The test statistic is

$$LM = N \cdot R^2 \sim \chi^2(p)$$
Test for ARCH Effects: LM Test

Null Hypothesis: no ARCH effects

Test Statistics:

    FORD
Test Stat 112.6884
  p.value  0.0000

Dist. under Null: chi-square with 12 degrees of freedom
  Total Observ.: 2000
A practical problem with this model is that, with $m$ increasing, the estimation of coefficients often leads to the violation of the non-negativity of the $\alpha$'s coefficients that are needed to ensure that conditional variance $\sigma_t$ is always positive. A natural way to achieve positiveness of the conditional variance is to rewrite an ARCH ($m$) model as

$$\sigma_t^2 = \alpha_0 + \mathbf{E}_{m,t-1}^T \Omega \mathbf{E}_{m,t-1}, \mathbf{E}_{m,t-1} = (\varepsilon_{t-1}, \ldots, \varepsilon_{t-m})^T$$
Forecasts of the ARCH model can be obtained recursively. Consider an ARCH\((m)\) model. At forecast origin \(h\), the 1-step ahead forecast of \(\sigma_{h+1}^2\) is

\[
\sigma_h^2 (1) = \alpha_0 + \alpha_1 \varepsilon_h^2 + \cdots + \alpha_m \varepsilon_{h+1-m}^2
\]

The 2-step ahead forecast of \(\sigma_{h+2}^2\) is

\[
\sigma_h^2 (2) = \alpha_0 + \alpha_1 \sigma_h^2 (1) + \alpha_2 \varepsilon_h^2 + \cdots + \alpha_m \varepsilon_{h+2-m}^2
\]

The \(l\)-step ahead forecast of \(\sigma_{h+l}^2\) is

\[
\sigma_h^2 (l) = \alpha_0 + \sum_{i=1}^{m} \alpha_i \sigma_h^2 (l-i)
\]

where

\[
\sigma_h^2 (l-i) = \varepsilon_{h+l-i}^2 \quad \text{if} \quad l-i \leq 0
\]
Weakness of ARCH Models

• The model assumes that positive and negative shocks have the same effects on volatility because it depends on the square of the previous shocks.

• The ARCH model is a little bit restrictive because for a finite fourth moment is necessary to be \( \alpha_t^2 \in [0, 1/3) \). This constraint becomes complicated for higher order of the ARCH.

• The ARCH model provides only way to describe the behavior of the conditional variance. It gives no any new insight for understanding the source of variations of a financial time series.

• ARCH models are likely to overpredict the volatility because they respond slowly to large isolated shocks.
Although the ARCH model is simple, it often requires many parameters to adequately describe the volatility process. To obtain more flexibility, the ARCH model was generalised to GARCH model. This model works with the conditional variance function.
GARCH model

\[ r_t = \mu_t + \varepsilon_t, \quad \varepsilon_t = \sigma_t \cdot \eta_t \]

\[ \sigma_t^2 = \alpha_0 + \sum_{i=1}^{m} \alpha_i \varepsilon_{t-i}^2 + \sum_{j=1}^{s} \beta_j \sigma_{t-j}^2 \]

\[ \eta_t, E(\eta_t) = 0, Var(\eta_t) = 1 \]

\[ \alpha_0 > 0, \alpha_i \geq 0, \beta_j \geq 0, \]

\[ \max(m,s) \]

\[ \sum_{i=1}^{\max(m,s)} (\alpha_i + \beta_i) < 1, \quad \text{and} \quad \alpha_i = 0 \quad \text{for} \quad i > m, \]

\[ \beta_j = 0 \quad \text{for} \quad j > s \]
A Connection to the ARMA process

For better understanding the GARCH model is useful to use the following transform:

\[ \xi_t = \varepsilon_t^2 - \sigma_t^2 \] so that \[ \sigma_t^2 = \varepsilon_t^2 - \xi_t \], and \[ \sigma_{t-i}^2 = \varepsilon_{t-i}^2 - \xi_{t-i} \].

Substitute these expressions into GARCH equation and we get as follows

\[ \varepsilon_t^2 = \alpha_0 + \sum_{i=1}^{\max(m,s)} (\alpha_i + \beta_i) \varepsilon_{t-i}^2 + \xi_t - \sum_{j=1}^{s} \beta_j \xi_{t-j} \]

The process \( \{\xi_t\} \) is a martingale difference series, i.e.,

\[ E\xi_t = 0, \text{ cov}(\xi_t, \xi_{t-j}) = 0, \text{ for } j \geq 1. \]
GARCH(max(m,s),s)

The process

\[ \varepsilon_t^2 = \alpha_0 + \sum_{i=1}^{\max(m,s)} (\alpha_i + \beta_i) \varepsilon_{t-i}^2 + \xi_t - \sum_{j=1}^{s} \beta_j \xi_{t-j} \]

is an application of the ARMA idea to the squared series \( \varepsilon_t^2 \). Thus

\[ E\varepsilon_t^2 = \alpha_0 / \left( 1 - \sum_{i=1}^{\max(m,s)} (\alpha_i + \beta_i) \right) \]
**GARCH(1,1)**

$$\sigma_t^2 = \alpha_0 + \alpha_1 \varepsilon_{t-1}^2 + \beta_1 \sigma_{t-1}^2$$

$$\alpha_0 > 0, \ 1 > \alpha_1 \geq 0, \ 1 > \beta_1 \geq 0, \ (\alpha_1 + \beta_1) < 1$$

- A large $\varepsilon_{t-1}^2$ or $\sigma_{t-1}^2$ gives rise to a large $\sigma_t^2$.
- A large $\varepsilon_{t-1}^2$ tends to be followed by large $\varepsilon_t^2$ generating the well-known behavior of volatility clustering in financial time series.
- If $1 - 2\varepsilon_{t-1}^2 - (\alpha_1 + \beta_1)^2 > 0$ then kurtosis
  $$3 \left(1 - (\alpha_1 + \beta_1)^2\right) / 1 - 2\varepsilon_{t-1}^2 - (\alpha_1 + \beta_1)^2 > 3$$
- The model provides a simple parametric function for describing the volatility evolution.
GARCH(1,1)

returns

$r_n$

$n$

time

\[-1.229\]
Forecasting by GARCH(1,1)

We assume that the forecast origin is $h$.

For 1-step ahead forecast

$$\sigma_h^2(1) = \alpha_0$$

$$+ \alpha_1 \varepsilon_h^2 + \beta_1 \sigma_h^2$$

For 2-step ahead forecast

$$\sigma_h^2(2) = \alpha_0 + (\alpha_1 + \beta_1) \sigma_h^2(1)$$

For $l$-step ahead forecast

$$\sigma_h^2(l) = \frac{\alpha_0 \left(1 - (\alpha_1 + \beta_1)^{l-1}\right)}{1 - \alpha_1 - \beta_1} + (\alpha_1 + \beta_1)^{l-1} \sigma_h^2(1)$$

Therefore

$$\sigma_h^2(l) \xrightarrow[l \to \infty]{} \frac{\alpha_0}{1 - \alpha_1 - \beta_1}$$
The multistep ahead volatility forecasts of GARCH(1,1) model converge to the unconditional variance of $\varepsilon_t$ as the forecast horizon increases to infinity provided that $\text{Var}(\varepsilon_t)$ exists.
GARCH(1,1) Forecasting

returns

- $r_n$
- $r_{nn}$
- $r_{nnn}$
- $r_{nnn}$

- original realization
- forecasting
- +standard deviation
- -standard deviation

Time
Mean Equation: ford.s ~ 1

Conditional Variance Equation:  ~ garch(1, 1)

Coefficients:

C 7.708e-004
A 6.534e-006
ARCH(1) 7.454e-002
GARCH(1) 9.102e-001
Series and Conditional SD

Original Series

Conditional SD

Values


Q2 Q3 Q4 Q1 Q2 Q3 Q4 Q1 Q2 Q3 Q4 Q1 Q2 Q3 Q4 Q1 Q2 Q3 Q4 Q1 Q2 Q3 Q4 Q1 Q2 Q3 Q4 Q1 Q2 Q3 Q4 Q1 Q2 Q3 Q4 Q1
Series with 2 Conditional SD Superimposed
ACF of Observations

Lags
ACF of Squared Observations

Lags
GARCH Volatility

Conditional SD

volatility
GARCH Standardized Residuals

residuals

Standardized Residuals

Q2 Q4 Q2 Q4 Q2 Q4 Q2 Q4 Q2 Q4 Q2 Q4 Q2 Q4 Q2 Q4

ACF of Std. Residuals

ACF

Lags

0 10 20 30

0.0 0.2 0.4 0.6 0.8 1.0
ACF of Squared Std. Residuals
QQ-Plot of Standardized Residuals

Quantiles of gaussian distribution

Standardized Residuals

10/13/1989
10/19/1987
12/23/1991
garch(formula.mean = ford.s ~ 1, formula.var = ~ garch(1,1))
Mean Equation: ford.s ~ 1
Conditional Variance Equation: ~ garch(1, 1)
Conditional Distribution: gaussian

---

**Estimated Coefficients:**

|            | Value     | Std.Error  | t value | Pr(>|t|) |
|------------|-----------|------------|---------|----------|
| C          | 7.708e-004| 3.763e-004 | 2.049   | 0.02031225 |
| A          | 6.534e-006| 1.745e-006 | 3.744   | 0.00009313 |
| ARCH(1)    | 7.454e-002| 5.362e-003 | 13.902  | 0.00     |
| GARCH(1)   | 9.102e-001| 8.762e-003 | 103.883 | 0.00     |
AIC(4) = -10503.79  
BIC(4) = -10481.39

Normality Test:

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<th>P-value</th>
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<td></td>
<td>P-value</td>
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<td>Shapiro-Wilk</td>
<td>0.9915</td>
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<td></td>
<td>P-value</td>
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Ljung-Box test for standardized residuals:

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<th>Chi^2-d.f.</th>
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<td>12</td>
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Ljung-Box test for squared standardized residuals:

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<td>14.04</td>
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Lagrange multiplier test:

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<th>F-stat</th>
<th>P-value</th>
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<td>14.77</td>
<td>0.2545</td>
<td>1.352</td>
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Test for Residual Autocorrelation

Test for Autocorrelation: Ljung-Box

Null Hypothesis: no autocorrelation

Test Statistics:
Test Stat  14.8161
p.value    0.2516
Test for Residual^2 Autocorrelation

Test for Autocorrelation: Ljung-Box

Null Hypothesis: no autocorrelation

Test Statistics:
Test Stat       14.0361
p.value          0.2984
GARCH(1,1)
GARCH(1,1) Forecasting

returns

- original realization
- forecasting
- +standard deviation
- -standard deviation
Weakness of GARCH Models

This model encounters the same weakness as the ARCH model. In addition, recent empirical studies of high-frequency financial time series indicate that the tail behavior of GARCH models remains too short even with standardised Student $t$-innovations.
ARCH-M model

The return of a security may depend on its volatility. If we take conditional deviation as a measure for risk, it is possible to use risk as a regressor in returns modeling. To model such a phenomenon, one may consider the ARCH-M model, where M means in mean. This type model was introduced by Engle in the paper

A simple ARCH-M(1) model

\[ r_t = \mu_t + c\sigma_t \cdot \eta_t \]

\[ \sigma_t^2 = \alpha_0 + \alpha_1 \varepsilon_{t-1}^2 + \cdots + \alpha_m \varepsilon_{t-m}^2 \]

The parameter $c$ is called the risk premium parameter. A positive $c$ indicates that the return is positively related to its past volatility.
GARCH-M Model

A simple GARCH(1,1)-M model

\[ r_t = \mu_t + c \cdot \sigma_t^2 + \varepsilon_t \]

\[ \varepsilon_t = \sigma_t \cdot \eta_t \]

\[ \sigma_t^2 = \alpha_0 + \alpha_1 \varepsilon_{t-1}^2 + \beta_1 \sigma_{t-1}^2 \]

The parameter \( c \) is called the risk premium parameter. A positive \( c \) indicates that return is positively related to its past volatility. By formulation we can see that there are serial correlations in the return series \( r_t \).
GARCH-M(1,1)

returns $r_n$

Back
IGARCH

IGARCH models are unit-root (integrated) GARCH models. This type model was introduced by Engle and Bollerslev in the paper “Modelling the Persistence of Conditional Variances,” *Econometric Reviews*, 5, (1986), pp. 1-50.

A key feature of IGARCH models is that the impact of past squared shocks $\zeta_{t-i} = \varepsilon^2_{t-i} - \sigma^2_{t-i}$ for $i > 0$ on $\varepsilon_t$ is persistent.
IGARCH(1,1)

An IGARCH(1,1) model can be written as

\[ r_t = \mu_t + \varepsilon_t, \quad \varepsilon_t = \sigma_t \eta_t, \]

\[ \sigma_t^2 = \alpha_0 + \beta_1 \sigma_{t-1}^2 + \alpha_1 \varepsilon_{t-1}^2 \]

where \( \{\varepsilon_t\} \) is defined as before and \( 1 > \beta_1 > 0 \).
Forecasting by IGARCH(1,1)

When \( \alpha_1 + \beta_1 = 1 \) repeated substitutions in

\[
\sigma_h^2 (l) = \alpha_0 + (\alpha_1 + \beta_1) \sigma_h^2 (l - 1)
\]

give \( \sigma_h^2 (\ell) = \sigma_h^2 (1) + (\ell - 1) \alpha_0, \quad \ell \geq 1, \)

where \( h \) is the forecast origin. The effect of \( \sigma_h^2 (1) \) on future volatilities is also persistent, and the volatility forecasts form a straight line with slope \( \alpha_0 \). The case of \( \alpha_0 = 0 \) is of particular interest in studying the IGARCH(1,1) model. The volatility forecasts are simply \( \sigma_h^2 (1) \) for all forecast horizons.
IGARCH(1,1) Forecasting

returns

- $r_n$
- $r_{nn}$
- $r_{rrn}$
- $r_{rrrn}$

$n, nn$

time

- $31.071$
- $6.151$

- $1.1 \times 10^3$
FARIMA

General IGARCH(p,q) process is called FARIMA(p,d,q):

\[ r_t = \mu_t + \varepsilon_t, \quad \varepsilon_t = \sigma_t \eta_t, \]

\[ \sigma_t^2 = \alpha_0 + \beta_1 (\sigma_{t-1}^2 - \varepsilon_{t-1}^2) + (1 + \alpha_1)\varepsilon_{t-1}^2 + (\alpha_1 + \beta_1)\varepsilon_{t-2}^2 \]
EGARCH Model

This model allows to consider asymmetric effects between positive and negative asset returns through the weighted innovation

\[ g(\eta_t) = \theta \eta_t + \gamma \left[ |\eta_t| - E(|\eta_t|) \right] \]

where \( \theta \) and \( \gamma \) are real constant.
Leverage Effects

Negative shocks tend to have a larger impact on volatility than positive shocks. Negative shocks tend to drive down the shock price, thus increasing the leverage (i.e. the debt-equity ratio) of the shock and causing the shock to be more volatile. The asymmetric news impact is usually referred to as the leverage effect.
Because $|\eta_t| - E(|\eta_t|)$ is iid sequence then $E\left[g(\eta_t)\right] = 0.$

So $\text{EGARCH}(m,s)$ can be written as

\[
\sigma_t^2 = \exp\left(\alpha_0 + \frac{1 + \beta_1 L + \cdots + \beta_s L^s}{1 - \alpha_1 L - \cdots - \alpha_m L^m} g(\eta_{t-1})\right)
\]

\[
r_t = \mu_t + \varepsilon_t, \quad \varepsilon_t = \sigma_t \cdot \eta_t
\]
The asymmetry of $g(\eta_t)$ can easily be seen as

$$g(\eta_t) = \begin{cases} 
(\theta + \gamma)\eta_t - \gamma E(|\eta_t|) & \text{if } \eta_t \geq 0 \\
(\theta - \gamma)\eta_t - \gamma E(|\eta_t|) & \text{if } \eta_t < 0
\end{cases}$$
and

\[
\sigma_t^2 = \sigma_{t-1}^2 \exp \left[ \alpha_0 (1 - \alpha_1) - \gamma \sqrt{2 / \pi} \right] \begin{cases} 
\exp \left[ (\theta + \gamma) \frac{\varepsilon_{t-1}}{\sqrt{\sigma_{t-1}^2}} \right] & \text{if } \varepsilon_{t-1} \geq 0 \\
\exp \left[ (\theta - \gamma) \frac{\varepsilon_{t-1}}{\sqrt{\sigma_{t-1}^2}} \right] & \text{if } \varepsilon_{t-1} < 0
\end{cases}
\]
garch(formula.mean = hp.s ~ 1, formula.var = ~
egarch(1,1), leverage = T, trace = F)

Mean Equation: hp.s ~ 1
Conditional Variance Equation: ~ egarch(1, 1)

Coefficients:

C  0.000313
A  -1.037907
ARCH(1)  0.227878
GARCH(1)  0.886652
LEV(1)  -0.133998
CHARMA

This model uses random coefficients to produce a conditional heteroscedasticity. The CHARMA model has the following form

\[ r_t = \mu_t + \varepsilon_t, \]

\[ \varepsilon_t = \delta_{1t} \varepsilon_{t-1} + \delta_{2t} \varepsilon_{t-2} + \ldots + \delta_{mt} \varepsilon_{t-m} + \eta_t \]

where \( \eta_t \sim \mathcal{N}(0, \sigma_\eta^2) \), \( \{ \delta_t \} = \left\{ (\delta_{1t}, \ldots, \delta_{mt})^T \right\} \)

is a sequence of iid random vectors with mean zero and non-negative definite covariance matrix \( \Omega \) and \( \{ \delta_t \} \) is independent of \( \{ \eta_t \} \)
The CHARMA model can easily be generalized so that the volatility of $\varepsilon_t$ may depend on some explanatory variables. Let $\{x_{it}\}_{t=1}^m$ be $m$ explanatory variables available at time $t$. Consider the model

$$r_t = \mu_t + \varepsilon_t, \quad \varepsilon_t = \sum_{i=1}^m \delta_{it} x_{i,t-1} + \eta_t$$

where $\{\delta_t\} = \left(\delta_{lt},...,\delta_{mt}\right)^T$ and $\{\eta_t\}$ are sequences of random vectors and random variables.
returns

$\tau_{-3.262}$

$\tau_{11.25}$

time

CHARMA(2)
Then the conditional variance of $\mathcal{E}_t$ is

$$\sigma_t^2 = \sigma_\eta^2 + \left( x_{1,t-1}, \ldots, x_{m,t-1} \right) \Omega \left( x_{1,t-1}, \ldots, x_{m,t-1} \right)^T$$

where $\Omega$ is a diagonal matrix
Stochastic Volatility Model

\[ r_t = \mu_t + \varepsilon_t, \quad \varepsilon_t = g\left(\sigma_t^2\right)\eta_t, \]
\[ \sigma_t^2 = \alpha_0 + \alpha_1 \sigma_{t-1}^2 + \nu_t \]

where
\[ \{\eta_t\} \sim iid \left(0,1\right) \]
\[ \{\nu_t\} \sim iid \left(0,\sigma_\nu^2\right) \]
\[ \{\eta_t\} \perp \{\nu_t\} \]
The Routes Ahead

• High frequency volatility
• High dimension correlation
• Derivative pricing
• Modeling non-negative processes
• Analysing conditional simulations by Least Squares Monte Carlo
Thank you very much for your attention