Definitions and Conventions

- **Asset Returns**

  We basically consider *Asset Returns* instead of *Prices* because of

  1) size of investment does not affect price changes as for average investor, market is perfectly competitive

  2) theoretical and empirical reasons - stationarity and erodicity (time average of quantity converges to its expectation, satisfied by i.i.d. observations)

  Let $P_t$ be the price of an asset in time $t$ and let us assume, that asset does not pay dividends. Then,

  $$ R_t = \frac{P_t - P_{t-1}}{P_{t-1}} = \frac{P_t}{P_{t-1}} - 1, $$

  is *Simple net Return*, or *Rate of Return*. Morover, $1 + R_t$ is called *Simple Gross Return*.

- **Real-World Examples**

  ![Prices of Microsoft 2000-2007](image)
Simple Gross Return over k-periods, or multiperiod, or Compound Return is defined by

\[ 1 + R_t(k) = (1 + R_t) (1 + R_{t-1}) \ldots (1 + R_{t-k+1}) = \frac{P_t}{P_{t-k}} \]

Annualized returns are defined as

\[ \text{Ann}[R_t(k)] = \left( \prod_{j=0}^{k-1} (1 + R_{t-j}) \right)^\frac{1}{k} - 1 \approx \frac{1}{k} \sum_{j=0}^{k-1} R_{t-j} \]
Continuous Compounding

**Continuously Compounded Return**, or **Log Return** is defined as:

\[ r_t = \log(1 + R_t) = \log \frac{P_t}{P_{t-1}} = p_t - p_{t-1}, \]

where \( P_t \) is price of an asset in time \( t \) and \( p_t = \log P_t \).

It is easier to manipulate with continuously compounded returns, and it has also some advantageous properties. One of them is, that multi-period return is sum of continuously compounded single-period returns:

\[ r_k = \log(1 + R_k) = \log((1 + R_t) \ldots (1 + R_{t-k+1})) = r_t + r_{t-1} + \ldots + r_{t-k+1}. \]

Disadvantage is that simple return on portfolio is a weighted average of the simple returns, but continuously compounded returns are just approximated with weighted average, as log of sum is not the same as the sum of logs and \( r_p = \sum_{i=1}^{N} w_i r_{i,k} \) where \( w \) is weight of \( i \)-th set in portfolio. Instead we can use simple returns with cross-section of assets.

**Dividend Payment and Excess Returns**

Assume regular dividend payment \( D_t \) at time \( t \) paid before date \( t \) price \( P_t \) (\( P_t \) is always ex-dividend). Then simple return at \( t \) is:

\[ R_t = \frac{P_t + D_t}{P_{t-1}} - 1, \text{ or } r_t = \log(P_t + D_t) - \log(P_t) \]

**Distribution of Returns**

As asset prices are highly correlated in time, we study behavior of asset returns. Perhaps most important characteristics of asset returns is their randomness. Thus **uncertainty** of financial time series is most important concept in financial modeling.

**Joint Distribution**

For a collection of \( N \) assets at time \( t \) with returns \( R_{it} \) in time \( t = 1, \ldots, T \), the **joint distribution function** is defined as:

\[ G(R_{1t}, \ldots, R_{N1}; R_{1t}, \ldots, R_{N2}; \ldots; R_{1T}, \ldots, R_{NT}; x | \theta), \]

where \( x \) is a vector of **state variables**, i.e. variables that summarize economic environment in which the asset returns are determined, \( \theta \) is a vector of fixed parameters that uniquely determine \( G \). Thus probability law \( G \) governs the stochastic behavior of asset returns and \( x \).

Of course this definition is far too general to be useful in testing, but it is useful to define and organize many models. If we deal with cross section of returns at a single date \( t \), we should assert that returns are statistically independent through time and that the joint distribution is identical across time, which yields strong implications for asset pricing models.
- The Conditional Distribution

Joint distribution $F$ of $\{R_i, \ldots, R_{iT}\}$ for given asset $i$ can be always rewritten as:

$$F(R_i, \ldots, R_{iT}) = F_{i1}(R_{i1}).F_{i2}(R_{i2} | R_{i1}).F_{i3}(R_{i3} | R_{i2}, R_{i1}) \ldots F_{iT}(R_{iT} | R_{iT-1}, \ldots, R_{i1}),$$

which is very useful in predictability of asset returns, and empirical estimations.

- The Unconditional Distribution (Normal Distribution)

Unconditional distribution may be of interest in case, when the expected predictability is minimal. Traditional assumption in many models is, that series are identically and independently distributed - IID, with fixed first moment - mean and second moment - variance, thus normally distributed. If series $x$ are normally distributed, i.e. $x \sim N(\mu, \sigma^2)$, then the Probability Density Function (PDF) is:

$$f(x) = \frac{1}{\sqrt{2\pi \sigma^2}} e^{-\frac{1}{2} \left( \frac{x-\mu}{\sigma} \right)^2}.$$ 

This is a "bell-shaped" curve centered on $\mu$.

IID assumption is tractable, but has important drawbacks: 1) Simple return has lower bound $-1$, but normal distribution may assume any real number. 2) if simple return is normally distributed, then multiperiod return is not normally distributed, because it is product of one-period returns. 3) Finally, empirical tests suggest, that returns tend to have positive excess kurtosis and "fatter tails" - are leptokurtic, and does not have fixed variance.

- Moments of Distribution

Mean $\mu$ (First moment)

Variance $\sigma^2$ (Second moment)

Skewness (Third moment) - $\frac{E[(X-\mu)^3]}{\sigma^3}$, distributions are said to be positively or negatively skewed (see example below).

$N(\mu, \sigma^2)$ is symetric

Kurtosis (Fourth moment) - $\frac{E[(X-\mu)^4]}{\sigma^4}$, normal random variable $N(\mu, \sigma^2)$ has kurtosis 3, and is sometimes called mesokurtic, distributions with higher kurtosis are said to be leptokurtic, distributions with kurtosis less than 3 are said to be platykurtic, (see example below). Most of the financial data exhibit leptokurtic distribution, thus have higher peak, and "fatter tails"
Examples of departures from Normal Distribution

- Normal Distribution

![Normal Distribution Graph](QF_I_Lecture2.nb)
- Positively Skewed Distribution

- Negatively Skewed Distribution
- **Leptokurtic Distribution**

![Leptokurtic Distribution Diagram]

- **Platykurtic Distribution**

![Platykurtic Distribution Diagram]
### The Lognormal Distribution

Alternative to normal distribution is to assume that continuously compounded single-period returns $r_t$ are IID normal, which implies that single-period gross simple returns are distributed as IID lognormal, since $r_t = \log(1 + R_t) \sim N(\mu, \sigma^2)$. Thus

$$\ln P_t \sim N[\ln P_i + (\mu - \frac{\sigma^2}{2}) (T-t), \sigma^2 (T-t)]$$

Simple returns are hence lognormal variables with mean $E(R_t) = e^{\mu + \frac{\sigma^2}{2}} - 1$ and variance $\text{Var}(R_t) = e^{2\mu + \sigma^2} \left( 1 + e^{\sigma^2} \right)$

Assumption of LogNormal distribution of returns is not consistent with all properties of empirical returns data, but adds advantage of not violating limited liability, since liability yields a lower bound of zero when $r_t$ is assumed to be normal.

![Lognormal Distribution](image.png)

### Stable Levy distribution

Levy $\alpha$-stable distribution has important property of stability: If a number of i.i.d random variables have a stable distribution, then a linear combination of these variables will have the same distribution, except for possibly different shift and scale parameters.

Normal Distribution, and Cauchy distribution are special cases of stable distribution (see example below)

Levy distribution departures from normality in similar way as financial data, thus it is more suitable for their analysis. For all $\alpha \neq 2$, stable distributions are heavy-tailed.

Levy $\alpha$-stable distribution has PDF characterized by 4 parameters - $\mu$ and $c$ are shift and scale parameters, which do not determine the shape of distribution, $\alpha$ exponent and $\beta$ measure of asymmetry. PDF is of form:

$$f(x; \alpha, \beta, c, \mu) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(i \mu - |ct|^\alpha (1 - i \beta \text{sgn}(t) \Phi) \exp(-itx) \, dt,$$ where $\Phi = \tan(\frac{\pi \alpha}{2})$ for $\alpha \neq 1$, and $\Phi = -(\frac{2}{\pi}) \log |t|$ for $\alpha = 1$. 
- Set $a = 2$ for Gaussian distribution with variance $\sigma^2 = 2c^2$ and mean $\mu$. Skewness parameter $\beta$ has no effect, set $c = 0.701$ for N(0,1) distribution
- Set $a = 1$ and $\beta = 0$ to get Cauchy distribution
- Set $a = 1/2$ and $\beta = 1$ to get Lévy distribution

- Stable analysis of Dow Jones Industrial Average

Here is an example, where Dow Jones industrial Average (DIA) returns from 1.1.2000 to today are considered. Histogram of the returns is contrasted with stable distribution fit with parameters: ($a = 1.59547$, $\beta = -0.0915485$, $c = 0.00560951$, $\mu = 0.000236478$). Notice, how well the theoretical distribution fits the data.
Market Information and Efficiency

Information is most important determination of success in finance. Efficiency simply means that informationally efficient market price changes must be unforecastable if they are properly anticipated, i.e. if they fully incorporate expectations and information of all participants. A market in which prices always fully reflect available information is called efficient.

Let \( P_t \) be a random variable defined on a filtered probability space \( (\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P}) \), which is also called stochastic basis, where \( \Omega \) is space of outcomes, \( \mathcal{F} \) is \( \sigma \) - algebra of the subsets of \( \Omega \), and \( \mathbb{P} \) is a probability measure on \( \mathcal{F} \) and \( (\mathcal{F}_t) \) is the usual filtration. A conditional probability \( \mathbb{P}[P_{t+1} | \mathcal{F}_t] \) is conditional probability of the set \( P_t \) being evaluated with the information available in the \( \sigma \) - algebra \( \mathcal{F} \).

- **Efficient Market Hypothesis**

  **Weak-form Efficiency**: The information set includes only the history of the prices or returns themselves. In other words, technical analysis is of no use. (Technical analysis is based on creating various basic indicators as trend-lines, support and resistance, volatility, momentum indicators etc. from past prices and volume. Indicators are used to produce trading (buy/sell) signals or rules. This is done mainly graphically by comparing the price and a trading rule.)

  **Semistrong-form Efficiency**: The information set includes all publicly available information known to all market participants. In other words, fundamental analysis is of no use. (Technical analysis is based on creating various basic indicators as trend-lines, support and resistance, volatility, momentum indicators etc. from past prices and volume. Indicators are used to produce trading (buy/sell) signals or rules. This is done mainly graphically by comparing the price and a trading rule.)

  **Strong-form Efficiency**: The information set includes all privately available information known to any market participant. In other words, even insider information is of no use.

**BUT** EHM is not testable, because of joint hypothesis problem, as any efficiency must assume an equilibrium model that defines normal returns, rejecting EHM implies that market is truly inefficient or incorrect model has been assumed, thus efficiency can never be rejected.
Random behavior of assets - Random Walk

One of the most important characteristics of returns is randomness. If the returns are random, then they can not be forecasted. Random Walk models are organized by various kinds of dependence between $r_t$ and $r_{t+k}$ at two dates, $t$ and $t+k$. Let $s$ define random variables $f(r_t)$ and $g(r_{t+k})$ where $f(\cdot)$ and $g(\cdot)$ are two arbitrary functions. All versions of RW are captured by orthogonality condition $\text{cov}[f(r_t), g(r_{t+k})] = 0$.

- Martingale

Earliest model of financial asset prices was martingale. By the means of a fair game, stochastic process $\{P_t\}_{t=0}^\infty$ satisfies condition $E[P_t | \mathcal{F}_t] = P_t$, where $P_t$ is stock price at time $t$ and is $\mathcal{F}_t$-measurable, $E[P_t] = \mathcal{F}_t$ are conditional expectations defined on the probabilistic space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$, where $\Omega$ is the space of market situations, $\mathcal{F}$ is $\sigma$-algebra of the subsets of $\Omega$. $\mathcal{F}_t$ is the usual filtration, $\mathcal{F}_t = \sigma(P_t, P_{t-1}, ..., P_0)$, which is also called information set, and $\mathbb{P}$ is a probability measure on $\mathcal{F}$. Then tomorrow price is expected to be equal to today price given the historical prices as information set.

Martingale hypothesis implies that the expected return is zero as: $E[P_{t+1} | \mathcal{F}_t] = P_t + E[r_{t+1} | \mathcal{F}_t]$. Then $E[r_{t+1} | \mathcal{F}_t] = 0$.

Martingale hypothesis implies that price changes are uncorrelated at all lags. Increments in value (changes in price) are unpredictable and conditional on the information set which is fully reflected in prices. Hence any attempt of linear and nonlinear forecasting rules is ineffective.

Martingale was considered to be an necessary condition for an EHM, main drawback is that it does not allow risk-expected return trade-off. If expected return was zero, none would invest.

Continuous-time martingale is a zero-drift stochastic process: $dp = \sigma dz$ where $dz$ is Wiener process and sigma is constant or other stochastic variable. If we rewrite martingale as $P_{t+1} = P_t + \varepsilon_t$, where $\{\varepsilon_t\}$ is a martingale difference sequence, RW is also fair game, but is stronger than martingale.

- Random Walk Hypothesis

A time series $\{p_t\}$ is a random walk process if:

$$p_t = \mu + p_{t-1} + \varepsilon_t,$$

where $p_t=\ln(P_t)$, thus $r_t = \mu + \varepsilon_t$, where $\mu$ is drift, and we distinguish between random walk with or without drift. Random walk without drift is special case of AR(1) process, with coefficient 1, thus it is not stationary. Hence, random walk model is unit-root nonstationary time series.

We distinguish between 3 forms of Random Walk:

RW1: $\varepsilon_t$ is independent and identically distributed $\sim$ iid, or $N \sim (0, \sigma^2)$, with conditional mean and variance $P_0 + \mu t$ and $\sigma^2 t$

RW2: $\varepsilon_t$ is independent, thus allows for heteroskedasticity. Test using filter rules, technical analysis

RW3: $\varepsilon_t$ is uncorrelated, thus allows for dependence in higher moments Test using autocorrelations, variance ratios, long horizon regressions.

RW is sufficient but not necessary condition for EHM. RW is nonstationary, conditional mean and variance are linear in time for all RW forms.
Example of Random Walk with drift $\mu$ and $\epsilon \in N(0, \sigma)$:

Initial settings $\mu=0$ (RW without drift), and $N(0,1)$ - Gaussian White Noise

Homework #1

Deadline: Monday 15.10.2007, 5 pm

*Homework may be returned in class, or sent via email to barunik@utia.cas.cz*

**:] Exercise 1 [:


**:] i [: Compute* a simple returns and log returns of all series

**:] ii [: Compute the sample mean, variance, skewness, excess kurtosis, minimum and maximum of all simple and log returns

**:] iii [: Plot Histograms of simple and log returns
Exercise 1:

i: Compute a simple returns and log returns of all series

ii: Compute the sample mean, variance, skewness, excess kurtosis, minimum and maximum of all simple and log returns

iii: Plot Histograms of simple and log returns

iv: Discuss the empirical characteristics of these series, discuss differences between statistics of simple returns and log returns, as well as differences between empirical distributions of HPQ, MSFT, AAPL, GOOG and stock index S&P 500

Exercise 2:
(bonus exercise)

Explain why you think the assumption of IID distribution of financial time series is or is not appropriate in financial econometrics.

* software used (any software, will be agreed in Class)