Abstract
The purpose of this article is to study a three-equations dynamic model. The task is to investigate the conditions of more complex behaviour of the model and its dependence on the model parameters. A linear model will be established at the first. A stability or non-stability of the model can be obtained employing Hurwitz theorem. Adding nonlinear perturbations in investment demand function we get nonlinear periodic oscillations. A bifurcation behavior of this system and its stability points are demonstrated.

Keywords: non-linear three-equation dynamic model, money market dynamics, uncovered interest rate parity, exchange rate dynamics, linearization, limit cycle

JEL classification: E44

Introduction
Considered model is based on the Mundell-Fleming theory that use IS-LM model for the description of commodity and money market. In the commodity market demand for investment and net export is related to savings. Adjustment processes cause the total demand and savings to be in equilibrium. The investment demand depends on interest rate and the net export demand is an increasing function of exchange rate. Popular Dornbusch model (Dornbusch, R. (1976)) built on the basement of Mundell-Fleming model is a dynamic model assuming price level and exchange rate as a characteristic variables of the model. The equilibrium in Dornbusch is saddle point equilibrium. Principles of the rational expectation are the reason why the system moves on the stable manifold. Turnovsky in his works (for example Turnovsky, S.(1997), Turnovsky, S. (2000)) also presents models of a small open economy but based on the assumption that a representative consumer chooses his equilibrium by solving inter-temporal optimization problem. Founded equilibrium is a saddle point.

Small-Open-Economy model presented in this article is based on the original Mundell-Fleming theory. Our interest will be focused on the dynamization of commodity and money market. Firstly we begin with an explanation of the linear dynamic model where economic nature of the parameters result in stability. A dynamics of the production $Y$ is described by the equation

$$\dot{Y} = \alpha \cdot [I(R, Y) + X(Y, Z) - S(Y, Z)]$$  \hspace{1cm} (1)

The expression in brackets of the equation (1) is the difference between the aggregate demand and supply. Remember that aggregate demand is $C + I + E$ where $C, I, E$ denote investment, consumption, demand for export respectively and aggregate supply is $C + S + M$ where $C, S, M$, is consumption, savings, import respectively. Subtracting aggregate supply from aggregate demand we get $I + X - S$ where $X = E - M$ denotes net export. The difference between aggregate demand and supply causes the increase of production if the difference is positive and decrease when it is negative. Dividing the equation (1) by $Y$ we get the following equation

$$\frac{\dot{Y}}{Y} = \alpha \left[ \frac{I(Y, R)}{Y} + \frac{X(Y, Z)}{Y} - \frac{S(Y, Z)}{Y} \right].$$  \hspace{1cm} (2)

Investment function could be consider in the form of

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\[ I = (i_0 - i_2 R - i_3 \log Z)Y \]  \hspace{1cm} (3)

where \(i_0 > 0, i_2 > 0\). We assume that net export depends on product \(Y\) and exchange rate \(Z\) in the following way

\[ X = (x_0 + x_3 \log Z)Y \]  \hspace{1cm} (4)

where \(x_0 > 0, x_3 > 0\). For savings \(S\) we assume

\[ S = (s_0 + s_1 \log Y + s_3 \log Z)Y \]  \hspace{1cm} (5)

where \(s_0 > 0, s_1 > 0\). A graph of the above function for \(s_0 = 0.2, s_1 = 0.1, s_3 = 0\) is presented in the figure 1.

Let denote the logarithm of variables \(Y, X\) and \(Z\) by lower case symbols i.e.

\[ y = \log Y, x = \log X, z = \log Z \]  \hspace{1cm} (6)

Using the relations (3)-(6) in the equation (2) we can get

\[ \dot{y} = \alpha[i_0 - i_2 R - i_3 z + x_0 + x_3 z - (s_0 + s_1 y + s_3 z)]. \]  \hspace{1cm} (7)

Money market is assumed to exhibit geometrical adjustment, i.e.

\[ e^{\dot{R}} = \left[ \frac{L(Y,R,Z)}{M} \right]^\beta, \beta > 0. \]  \hspace{1cm} (8)

\(M\) and \(L\) denotes money stock and demand for money respectively. The dependence of the demand for money on production, nominal interest rate and exchange rate is given by the following power function

\[ L(Y,R,Z) = L_0 Y^{l_1} (1 + R)^{-l_2} Z^{l_3}, \]  \hspace{1cm} (9)

where \(L_0, l_1, l_2\) are positive numbers. After making a logarithm of the equation (8) we get

\[ \dot{R} = \beta[l_0 + l_1 y - l_2 R - l_3 z - m]. \]  \hspace{1cm} (10)

As for the equation (10) remember that \(l_1\) is positive because of the assumption that the domestic money and foreign money are considered to be strong substitutes. Third equation constituting the model describes exchange rate dynamics and is based on uncovered interest rate parity. Uncovered interest rate parity is the following relation

\[ Z_{t+h} = Z_t \frac{1 + hR}{1 + hR^*}, \]  \hspace{1cm} (11)

where \(h\) is the increase of time \(R_t\) is domestic interest rate parity \(R_t^*\) is foreign interest rate in the time \(t\). Making logarithm, dividing by \(h\) and letting \(h \to 0\) we get

\[ \dot{z} = R - R^*. \]  \hspace{1cm} (11)

The model consists of the equations (7), (10), and (11). This system of equations has the following form

\[ \dot{y} = \alpha[i_0 - i_2 R + x_0 + x_3 z - (s_0 + s_1 y)] \]
\[
\dot{R} = \beta(l_0 + l_1 y - l_2 R + l_3 z - m)
\]
\[
\dot{z} = R - R^*.
\]
This model is linear and it is stable for each economically admissible value of parameter.

**Equilibrium of the linear model**

An equilibrium of dynamical systems is defined as the situation where the variables that are characteristic for the system does not move with time. So the system of equations describing the system looks as

\[
\begin{align*}
0 &= i_0 - i_2 \bar{R} - i_3 z + x_0 + x_3 \bar{z} - s_0 - s_1 \bar{y} - s_2 z, \\
0 &= l_0 + l_1 \bar{y} - l_2 \bar{R} + l_3 \bar{z} - m, \\
0 &= \bar{R} - R^*
\end{align*}
\]

where \(\bar{y}, \bar{R}\) and \(\bar{z}\) denotes equilibrium values of the variables of the model. An question if the equilibrium of the model will be stable or not will be solved using of well known Hurwitz’s rule. A Jacobian of the system of equations (12) is

\[
A = \begin{bmatrix}
-\alpha s_1 & -\alpha i_2 & \alpha (x_3 - i_3 - s_3) \\
\beta l_1 & -\beta l_2 & \beta l_3 \\
0 & 1 & 0
\end{bmatrix}
\]

Let \(j = x_3 - i_3 - s_3\). Eigenvalues of matrix \(A\) are the roots of the following equation

\[
\det(\lambda I - A) = \det \begin{bmatrix}
\lambda + \alpha s_1 & \alpha i_2 & -\alpha j \\
-\beta l_1 & \lambda + \beta l_2 & -\beta l_3 \\
0 & -1 & \lambda
\end{bmatrix} = 
\]

\[
= \lambda^3 + \alpha(s_1 + l_2)\lambda^2 + \beta(\alpha s_1 l_2 + \alpha i_2 l_1 - l_3)\lambda - \alpha\beta(l_3 s_1 + l_1 j) = 0.
\]

The coefficients of the characteristic equation are used for the construction of Hurwitz’s matrices.

\[
H_1 = \alpha(s_1 + l_2), \quad H_2 = \begin{bmatrix}
\alpha(s_1 + l_2) & 1 \\
-\alpha\beta(l_3 s_1 + l_1 j) & \beta(\alpha s_1 l_2 + \alpha i_2 l_1 - l_3)
\end{bmatrix},
\]

\[
H_3 = \begin{bmatrix}
\alpha(s_1 + l_2) & 1 & 0 \\
-\alpha\beta(l_3 s_1 + l_1 j) & \beta(\alpha s_2 s_1 + \alpha i_1 l_2 - l_3) & \alpha(s_1 + l_2) \\
0 & 0 & -\alpha\beta(l_3 s_1 + l_1 j)
\end{bmatrix}.
\]

Let us compute the determinants of \(H_i\), \(i=1, 2, 3\)

\[
\det H_1 = \alpha(s_1 + l_1),
\]

\[
\det H_2 = \alpha\beta(s_1 + l_2)(\alpha s_1 l_2 + \alpha i_2 l_1 - l_3) + \alpha\beta(l_3 s_1 + l_1 j),
\]

\[
\det H_3 = -\alpha\beta(l_3 s_1 + l_1 j)[\alpha\beta(s_1 + l_2)(\alpha s_1 l_2 + \alpha i_2 l_1 - l_3) + \alpha\beta(l_3 s_1 + l_1 j)].
\]

The expression \(j = x_3 - i_3 - s_3\) be interpreted as a sensitivity of the total demand for commodities on the exchange rate. Let us assume \(j<0\). This assumption corresponds with a reality. Such behaviour pattern which is a consequence of relatively low positive sensitivity \(x_3\) of export to the exchange rate and low negative sensitivity of the demand for imported consumers and investment goods to the exchange rate \(-i_3,s_3\) is observable in the economics indicators for the Czech Economy. Except for \(j\) all parameters of the system are assumed to be positive. Thus the expression

\[
l_3 s_1 + l_1 x_3
\]
can be both a positive and a negative. Let us assume that the expression (18) to be a negative. Then from the following equation
\[
\det H_3 = -\alpha \beta (l_3 s_1 + l_1 x_3) \det H_2
\]  
(19)
follows that
\[
\text{sign} \det H_2 = \text{sign} \det H_3.
\]  
(20)
If moreover \( \det H_2 > 0 \) then \( \det H_3 > 0 \). This implication corresponds to Hurwitz’s rule, i.e., the necessary and sufficient condition for a linear system to be stable is \( \det H_i > 0 \) for all \( i \). Thus a linear Small-Open-Economy model is stable for sufficiently small value \( j \).

**Example 1**

Let us introduce a numerical example. Let us use the following numerical values
\[
\begin{align*}
i_0 &= 0.048 & x_0 &= 0.26 & s_0 &= 0.1 & l_0 &= 0.25 \\
- &= - & s_1 &= 0.07 & l_1 &= 0.12 \\
i_2 &= 0.16 & - &= s_2 &= 0.16 & l_2 &= 0.6 \\
i_3 &= 0.8 & x_1 &= 0.3 & s_3 &= 0.1 & l_3 &= 0.1
\end{align*}
\]
An exogenous variable \( R^* \) is put equal 0.05, \( a=20, \ b=1, \) and \( m=0.48 \). Then the system (12) has the following form as follows
\[
\begin{align*}
\dot{y} &= 20 \cdot [0.211 - 0.07 y - 0.16 R - 0.6 z], \\
\dot{R} &= 1 \cdot [0.12 y - 0.6 R + 0.1 z - 0.23], \\
\dot{z} &= R - 0.05.
\end{align*}
\]  
(21)
Thus the system (21) is a linear system. The Jacobian of the system is the matrix
\[
A = \begin{bmatrix}
\alpha & y & R & z \\
\beta & y & R & z \\
0 & 1 & 0
\end{bmatrix}
\]

The determinant
\[
|A| = -1.3
\]
The eigenvalues
\[
eigA = \begin{pmatrix}
-1.779 \\
-0.111 + 0.848i \\
-0.111 - 0.848i
\end{pmatrix}
\]
A behaviour of the linear system is shown in the figure 2 where the time series production, interest rate, and exchange rate are shown.
Non-linear perturbation of the system

A non-linear dependence of investment and logarithm of the product is well-known. Therefore let us consider any non-linear relationship between investment and logarithm of the product. This non-linear relationship is formulated instead of the constant denoted by \( i_0 \). The dependence of this sort is possible represented by a logistic curve. This expression has the following form

\[
\frac{k}{1 + e^{a-y}}.
\]  

(22)

Having done it we get instead of (7)

\[
\dot{y} = \alpha \left[ \frac{k}{1 + e^{a-y}} - i_z R - i_y x_0 + x_1 z - (s_0 + s_1 y + s_2 z) \right].
\]  

(23)

Example 2

For this example we use the same numerical values as in the example 1 and moreover \( k=0.4 \), and \( a=4 \). The analysis of the non-linear system will be done through the following system

\[
\dot{y} = 20 \left[ -0.4 - 0.16 y - 0.07 y - 0.16 R - 0.6 z \right]
\]

\[
\dot{R} = 1 \left[ 0.12 y - 0.6 R + 0.1 z - 0.23 \right]
\]

\[
\dot{z} = R - 0.05.
\]

A Jacobian of the system (24), its determinant, and eigenvalues have the following form

\[
A = \begin{bmatrix}
\alpha(Dy - S_1) & \alpha(DR - S_2) & -\alpha S_3 \\
\beta y & \beta R & \beta z \\
0 & 1 & 0
\end{bmatrix}
\]

\[
A = \begin{bmatrix}
-0.556 & -3.2 & -12 \\
0.12 & -0.6 & 0.1 \\
0 & 1 & 0
\end{bmatrix}
\]

The determinant

\[
|A| = -1.384
\]

The eigenvalues

\[
eig A = \begin{bmatrix}
0.128 + 0.982i \\
0.128 - 0.982i \\
-1.412
\end{bmatrix}
\]

A behaviour of the non-linear system is shown in the figure 3 where the time series production, interest rate, and exchange rate are shown.
More general non-linear system

Let us consider more general non-linear system. This non-linear relationship between investment and discounted logarithm of the product can obtain by the following expression

\[ g(y(t)) = \frac{1}{1 + R} \cdot \frac{k}{1 + \exp(b - a \cdot y(t))} \]  

(25)

Having done it we can get

\[ \dot{y} = 20 \cdot \frac{1}{1 + R} \cdot \frac{0.4}{1 + \exp(1 - y(t))} - 0.16 - 0.07y - 0.16R - 0.6z \]

\[ \dot{R} = 1 \cdot [0.12y - 0.6R + 0.1z - 0.23] \]

(26)

\[ \dot{z} = R - 0.05 \]

Example 3

For analysis this system we again use the numerical values from previous examples. Moreover, let us put \( b = 1.0 \), and \( j0 = 0.1 \) of the non-linear system (26). The Jacobian of the system, its determinant, and eigenvalues have the following form

\[
A = \begin{bmatrix}
\alpha(Dy - S_1) & \alpha(DR - S_2) & -\alpha S_3 \\
\beta y & \beta R & \beta z \\
0 & 1 & 0
\end{bmatrix}
\]

\[
A = \begin{bmatrix}
-0.152 & -7.822 & -12 \\
0.12 & 0.6 & 0.1 \\
0 & 1 & 0
\end{bmatrix}
\]

The determinant

The eigenvalues
\[ |A| = -1.425 \]
\[ \text{eig} A = \begin{pmatrix} 0.628 + 1.17i \\ 0.628 - 1.17i \\ -0.808 \end{pmatrix} \]

A behaviour of this non-linear system is shown in the figure 4 where the time series production, interest rate, exchange rate, and phase portrait are shown.

**Figure 4**

**Conclusions**

The linear system of the Small Open Economy model with a negative sensitivity of aggregate demand on the exchange rate was investigated. Such system could be stable for sufficiently high \( a \) and for \( l_3 x_1 + l_2 x_2 < 0 \). Such stable linear system was demonstrated in the first part of the paper. Non-linear perturbations were be applied in the two cases. The first case exhibits limit cycle and the second one demonstrates non-periodic oscillations.

**References**


Guckenheimer, J. and Holmes, P.: *Non–Linear Oscillation, Dynamical Systems and Bifurcations of Vector Fields*. Springer Verlag, New York 1986


